**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part II. Optimization methods

## Lecture 15. Existence and uniqueness of the solution of optimization control problem

We continue to consider abstract optimization control problems. We will give the theorems of the existence and the uniqueness of its solution.

### 15.1. Uniqueness of the solution of optimization control problem

It should not be surprising that some extremum problems have unique so­lution and some don't. For example, the trivial problem of minimizing the simplest quadratic function

*f = f* (*x*) = *x*2

on the interval [-1,1] has a unique solution, while the problem of maximizing it has two solutions (see Figure 15.1).



Figure 15.1. The parabola has one minimum and two maxima in [-1,1]

The question is: What is the difference between the functions *f*(*x*)*=x2* and *g*(*x*)=- *x2* (maximizing *f* is equivalent to minimizing *g*)that causes one of them to have one minimum on [-1,1] and the other to have two? Evidently, the property of convexity is the key.

The function *f* = *f(x)* is said to be *convex* on the segment [*a,* *b*]if the following inequality holds:

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If this inequality turns into equality only if *x=y,* then *f(x)* is called *strictly convex.* Geometrically, the convexity of *f* means that the segment of the curve *f* = *f*(*x*)that connects the points (*x, f*(*x*))and (*y,f*(*y*))lies not higher (or, in the case of strict convexity, lower) than the segment of the straight line connecting these points (see Figure 15.2).



 a) strictly convex function b) non-convex function c) convex, but not strictly convex function

Figure 15.2. Convexity of functions.1

A simple example shows that strict convexity is required for the existence of a unique minimum (see Figure 15.3).



Figure 15.3. A minimum may be not unique if the function is not strictly convex

We can extend the definition of the convexity to an arbitrary functional on a convex set of a linear space

**Definition 15.1.** *A functional I defined on a* ***convex*** *set U is called convex if the following inequality holds:*

*I* [*σ**u* + (1–*σ*) *v*] ≤ *σ**I*(*u*)+ (1–*σ*) *I*(*v*) ∀*u*,*v*∈*U*, *σ*∈[0,1].

*The functional is* ***strict convex***, *if the last inequality is strict for*  *and* , .

**Theorem 15.1.** A *strictly convex functional defined on a* convex *set can* *have at most one point* *of minimum*

**Proof.** Suppose that a strictly convex functional *I* on a convex set *U* has two different points of minimum, *x* and *y.* Then the element belongs to *U* for any **. Since the functional is strictly convex, we have



The value of the functional at the chosen element of *U* is less than its minimum on *U.* The assumption that there are two points of minimum lead to a contradiction. □

### 15.2. Existence of the solution of optimization control problem

We present a result that states the shows the conditions of solvability for an extremum problem. Suppose that we need to find a function minimizing a functional *I* on a given set of admissible controls *U* of a space *V.* If *I* is lower bounded, then the image *I*(*U*)has a lower bound. This means that there exists a minimizing sequence, i.e., a sequence of elements such that *.* However, it is not known whether the sequence  itself converges.

Assume that *U* is a bounded subset of a normed vector space *V.* Then there exists a positive constant с such that  for all admissible control *v*. In this case, the sequence  is uniformly bounded, i. e.,  for all *k.* Let *V* be a Hilbert space, i.e., a complete unitary (space, where Cauchy criterion is true). By the Banach – Alaoglu theorem (a generalization of the classical Bolzano – Weierstrass theorem to the case of infinite-dimensional spaces),  has a subsequence that weakly converges in *V.* If we denotethe subsequence by  again, the weak convergence means that the scalar products  converge to  for every function *.*

We have established so far that there exists a weak limit of the minimizing sequence. But it is not clear if this limit belongs to the set of admissible controls. Suppose that *U* is convex and closed (and, consequently, contains the limits of every sequence of its elements that converges in the norm). It is known from the theory of Hilbert spaces that every convex closed subset I of a Hilbert space is weakly convex, i.e., it contains the limits of all weakly converging sequences of its elements. Since the minimizing sequence consists I only of the elements of *U* and weakly converges, it follows that its weak limit *и* belongs to *U* and is therefore an admissible control. However, it is not known whether the functional achieves its lower bound at *u.*

Assume that the functional *I* is convex and continuous. Every convex continuous functional is weakly lower semi continuous. This means that if weakly in *V,* then



This inequality implies that the sequence {*I*(*uk*)} has converging subsequences, although it does not necessarily converge itself As follows from the last inequality, since the functional is weakly semi continuous, *I*(*u*)does not exceed the lower bound of limits of all subsequences of {*I*(*uk*)}*.*

Since the subject of our consideration is not an arbitrary weakly converging sequence, but the one that minimizes the functional on *U,* {*I*(*uk*)} note only has converging subsequences, but converges itself to the lower bound of the functional *I* on *U.* Then the last inequality can be written in the form



which means that the value of the functional *I* at the element *и* does not exceed its lower bound on the set *U.* We established earlier that this element belongs to *U.* So the value of the functional cannot be less than . Then we obtain  We proof the existence of the admissible element such that the value of the given functional isequal to its lower bound on *U.* Thus, the admissible control *и* is a solution to the problem in question.

**Theorem 15.2.** *The problem of minimizing a convex lower semicontinuous* *functional bounded from below on a convex closed bounded subset of a Hilbert space is solvable.*